

Determinant and Weyl anomaly of Dirac operator: a holographic derivation

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Abstract. We present a holographic formula relating functional determinants: the fermion determinant in the one-loop effective action of bulk spinors in an asymptotically locally AdS background, and the determinant of the two-point function of the dual operator at the conformal boundary. The formula originates from AdS/CFT heuristics that map a quantum contribution in the bulk partition function to a subleading large- N contribution in the boundary partition function. We use this holographic picture to address questions in spectral theory and conformal geometry. As an instance, we compute the type-A Weyl anomaly and the determinant of the iterated Dirac operator on round spheres, express the latter in terms of Barnes' multiple gamma function and gain insight into a conjecture by Bär and Schopka.

1. Introduction

Ever since its appearance almost fifteen years ago in the form of Maldacena's conjecture, the AdS/CFT correspondence [1, 2, 3] has been a successful tool to address questions concerning strongly-coupled systems. Many developments depart from the original canonical formulation in pure anti-de Sitter (AdS) spacetime (mostly still restricted to classical (super)gravity in the bulk); for instance, finite-temperature effects on the boundary theory led to consideration of AdS black holes as bulk background geometries. This novel holographic approach, phenomenological in nature, covers an increasing amount of physical situations ranging from strongly coupled quark-gluon plasma and condensed matter systems to cosmological singularities and black hole physics. At present, this ambitious but conjectural program seems to succeed at a qualitative level (cf. [4]).

In contrast, quantitative and exact results in AdS/CFT correspondence are to be found in the interplay with mathematics; in for instance conformal geometry and spectral theory. Geometric roots of AdS/CFT date back to the seminal work of Fefferman and Graham [5, 6] that addresses conformal geometry on a compact manifold as geometry at the conformal infinity of space-filling Poincare metrics. Conversely, AdS/CFT revealed interesting conformal invariants, e.g. Q -curvature, that arise in the volume renormalization of these Poincare metrics and triggered new developments in conformal geometry (cf. [7, 8]).

The present contribution will focus precisely on these latter aspects of the duality, where the foreseeable progress seems modest but solid. We deal with 'holographic formulas' as special entries in the AdS/CFT dictionary, relating one-loop

determinants for bulk fields in asymptotically AdS backgrounds and determinants of correlation functions of the dual operators at the boundary. They originate in quantum refinements of the duality where one-loop corrections in the gravity side are mapped to sub-leading terms in the large-N expansion of the boundary theory. Interesting effects in for instance thermodynamics and transport phenomena on the boundary are captured by the holographic correspondence only after inclusion of quantum one-loop effects in the bulk (cf. [9, 10]). A systematic study of bulk *scalars* has led to a holographic formula which has been verified in certain cases amenable to analytic evaluation; these bulk geometries include pure and thermal AdS, the BTZ black hole, and other quotients or orbifolds of AdS [11, 12, 13, 14].

Our aim now is to show that also for bulk *spinors* an analogous holographic formula can be established. Explicit computations are performed for the ball model of hyperbolic space which embrace several scattered results in the literature. In this case the bulk side and the role of Barnes' gamma function, already explored in [15, 16], can be further exploited to get a closed formula for the determinant of the Dirac operator on round spheres, an interesting result that seems to have escaped notice. A universal formula for the type-A trace anomaly of Dirac operator is as well obtained in this holographic way. These explicit results contain previous ones found in relation with proposals for a c-theorem in dimensions other than two; they include Cardy's a-theorem [17, 18], universal terms in entanglement entropy [19, 20] and F-theorem [21].

We start in section 2 with a review of bulk spinors in AdS/CFT and its double quantization to predict an $O(1)$ quantum contribution to the partition functions. Next we analyze the dual picture at the boundary in section 3 in order to detect this $O(1)$ contribution to the partition function on the CFT side. In section 4 we write down the spinor holographic formula. In section 5, the pure AdS bulk geometry is considered as an instance where both sides of the formula can be worked out in detail. Section 6 is concerned with the Dirac operator at the boundary and the application of the holographic formula to read off the universal part in the associated Polyakov formulas, the type-A trace anomaly as well as the functional determinant on round spheres. In section 7 we examine several scattered results which are now encompassed by our calculations. Concluding remarks are given in section 8.

2. Bulk spinors and double quantization

The role of bulk spinors in AdS/CFT has, of course, been extensively studied since the early days of the correspondence; a non-exhaustive list includes [22, 23, 24, 25]. To begin with, we choose the bulk side and review the features relevant to our present concern, namely, the spinor version of the holographic formula.

Consider a bulk metric that approaches asymptotically the Poincaré half-space model for the Euclidean section of AdS_{n+1} , that is, hyperbolic space H_{n+1}

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} . \quad (1)$$

The solutions of Dirac equation $(\not{X} + m)\psi = 0$, with positive mass $m > 0$ for definiteness, behave near the conformal boundary $z = 0$ as[‡]

$$\psi \sim z^{\lambda_-} \psi_o(\vec{x}) + z^{\lambda_+} \chi_o(\vec{x}) , \quad (2)$$

[‡] Strictly speaking, valid for $0 < m < \frac{1}{2}$.

with $\lambda_{\pm} = \frac{n}{2} \pm m$, and the boundary data ψ_o and χ_o belong to the eigenspace of the flat Dirac gamma associated to the z -direction, Γ^o , with eigenvalues -1 and $+1$, respectively.

The requirement of regularity at the deep interior, $z \rightarrow \infty$, imposes a linear relation between ψ_o and χ_o given by convolution with the scattering operator $\chi_o = S(\lambda) * \psi_o$, or equivalently, with the kernel associated to the two-point function of the dual operator at the boundary. It is this on-shell relation which ultimately leads, upon functional differentiation of the action with respect to the boundary source, to the corresponding two-point correlator

$$\langle \mathcal{O}_+ \overline{\mathcal{O}}_+ \rangle \sim \frac{\vec{\Gamma} \cdot (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{1+n+2m}} . \quad (3)$$

The standard AdS/CFT recipe contemplates ψ_o as the source and χ_o as expectation value of the dual primary operator with conformal dimension λ_+ at the boundary. In the variational approach to the path integral, the sum over histories is preformed with ψ_o and $\bar{\psi}_o$ prescribed (q 's), whereas χ_o and $\bar{\chi}_o$ are free to vary (canonically conjugate momenta p 's). Nonetheless, a crucial observation (cf. [18, 21, 26]) is that whenever $0 < m < 1/2$ the dimension λ_- is above the unitarity bound, in this case one is free to interchange the roles of ψ_o and χ_o to obtain the two-point function of a dual operator of dimension λ_-

$$\langle \mathcal{O}_- \overline{\mathcal{O}}_- \rangle \sim \frac{\vec{\Gamma} \cdot (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{1+n-2m}} . \quad (4)$$

2.1. $O(1)$ contribution to the partition function

That is very much the picture at the classical level in the bulk. Considering now quantum fluctuations around the on-shell configuration, there are therefore two possible quantizations of the bulk spinor: the conventional one with ψ_o set to zero and another, alternative one, with χ_o set to zero. Whenever the mass of the bulk spinor lies in the window $0 < m < \frac{1}{2}$, both kinds of modes are normalizable. Their contribution to the partition function can be computed in the standard way via the Green function for the conventional modes (λ_+), and analytically continue to account for the alternative modes (λ_-). With this choice at hand, we can emulate the scalar case since now double quantization for bulk spinors is established. The relative change in the partition function, upon functional integration of the quantum fluctuations at quadratic order, is then given by the ratio of the associated functional determinants:

$$\frac{Z_{grav}^+}{Z_{grav}^-} = \frac{\det_+ \{ \not{\nabla}_x + m \}}{\det_- \{ \not{\nabla}_x + m \}} . \quad (5)$$

3. Boundary double-trace deformation

These two choices of asymptotic behavior correspond in AdS/CFT to two CFT's which share the same field content but differ in the dimension of the fermionic operator \mathcal{O} , dual to the bulk spinor. The UV CFT with \mathcal{O}_- , perturbed by the relevant double-trace deformation \mathcal{O}_-^2 , flows into the IR CFT with \mathcal{O}_+ (cf. [18, 21, 26]).

3.1. $O(1)$ contribution to the partition function: a shortcut

To get a handle on the relative change in the CFT partition functions at the end points of the RG flow, instead of considering the auxiliary field trick as in [18, 21], we simply adapt an argument given in [27] to relate Z_{UV} to Z_{IR} . Namely, in the path integral of the UV CFT we promote the sources η and $\bar{\eta}$ to dynamical fields and integrate over them

$$\frac{Z_{IR}}{Z_{UV}} = \int \mathcal{D}\bar{\eta} \mathcal{D}\eta \langle \exp \int (\bar{\eta} \mathcal{O}_- + \bar{\mathcal{O}}_- \eta) \rangle . \quad (6)$$

The expectation value can be approximated, at leading large N due to the factorization of the correlation functions, by

$$\exp \int \bar{\eta} \langle \mathcal{O}_- \bar{\mathcal{O}}_- \rangle \eta , \quad (7)$$

and the Gaussian integral results in the functional determinant of the two-point function

$$\frac{Z_{IR}}{Z_{UV}} = \det \langle \mathcal{O}_- \bar{\mathcal{O}}_- \rangle , \quad (8)$$

or, alternatively,

$$\frac{Z_{UV}}{Z_{IR}} = \det \langle \mathcal{O}_+ \bar{\mathcal{O}}_+ \rangle . \quad (9)$$

4. The holographic formula

We have now all necessary ingredients to write down the ‘spinor holographic formula’ that stems from the postulated equality of the partition functions in AdS/CFT correspondence, at subleading $O(1)$:

$$\frac{\det_- \{\not{X}_x + m\}}{\det_+ \{\not{X}_x + m\}} = \det \langle \mathcal{O}_\lambda \bar{\mathcal{O}}_\lambda \rangle_{\mathcal{M}} \quad (10)$$

The ‘+’ means that we compute with the standard λ in the asymptotic behavior of the bulk spinor, whereas ‘−’ means the analytic continuation $\lambda \rightarrow n - \lambda$. As usual, to make sense out of this formula one needs to tame the divergencies that arise due to the IR divergent volume of AdS and the UV divergent short distance singularities. Both divergencies turn out to be tied by the so-called IR/UV in AdS/CFT correspondence.

This formula conjecturally applies to bulk geometries X which are Euclidean sections of asymptotically locally AdS (ALAdS). In particular, when the conformal infinity \mathcal{M} belongs to the conformal class of the standard round spheres, and therefore conformally flat, the bulk is locally AdS and the IR-divergent volume of AdS factorizes. One can then read off an $O(1)$ contribution to the holographic trace anomaly in even n , just as in the case of a bulk scalar [3, 7, 28]. Alternatively, from the difference of one-loop effective actions one can compute the holographic type-A trace anomaly coefficient a following the general recipe of [29]. The behavior of this coefficient for even n and of a related quantity for odd n , in the cases we will explore, gives support to a conjectured c-theorem valid in all dimensions [20] § and to a F-theorem proposal [21] as well.

§ This promises to answer the disparity pointed out in [30] that, although the computation on the AdS side contemplates even and odd dimensions on equal footing, it is not clear how to translate the “holographic” central charge into field theory language in the case of odd-dimensional CFT’s.

5. The canonical case

As might be expected, the ball model for hyperbolic space (Euclidean AdS) turns out to be the simplest bulk background where calculations can be spelled out in detail and related to the CFT on the conformal boundary (the conformally flat class of the standard round sphere).

5.1. Bulk

The effective action for a Dirac spinor in hyperbolic space has been recently revisited [15, 16] in connection with a curious gauge-gravity duality where Barnes' multiple gamma function plays a central role. We briefly survey the relevant steps in the computation of the effective action

$$S_{grav}^+ = -\log \det\{\not{V} + m\} . \quad (11)$$

In terms of the Green's function, one has $(\not{V} + m)\mathcal{D} = -\mathbb{I}$,

$$S_{grav}^+ = \int^m \text{tr } \mathcal{D}^{(n+1)} , \quad (12)$$

where also the spinor indices are traced out. We refer to [15, 16] for details of the implementation of dimensional regularization and the nontrivial role of the bulk volume. In all, one gets the remarkable result, valid for both even and odd dimensions, in terms of Barnes' multiple gamma

$$\log \frac{\det_+ \{\not{V} + m\}}{\det_- \{\not{V} + m\}} = -2^{1+\lfloor \frac{n}{2} \rfloor} \cdot \log \frac{\Gamma_{n+1}(\frac{n+1}{2} + m)}{\Gamma_{n+1}(\frac{n+1}{2} - m)} . \quad (13)$$

In addition, whenever n is even one can read off the trace anomaly as in the scalar case [11, 12]. In the present case one essentially gets the integral of the spinor Plancherel|| measure(cf. [31]) times the volume anomaly $\mathcal{L}_{n+1} = 2(-\pi)^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$

$$\left[\frac{2}{(2\pi)^{\frac{n}{2}}} \int_0^m d\mu \frac{(\frac{1}{2} + \mu)^{\frac{n}{2}} \cdot (\frac{1}{2} - \mu)^{\frac{n}{2}}}{(\frac{1}{2})^{\frac{n}{2}}} \right] \cdot \mathcal{L}_{n+1} . \quad (14)$$

5.2. Boundary

For the round n -sphere as conformal boundary, the knowledge of the eigenvalues of the two-point function $\langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n}$ and their degeneracies allows a brute-force computation of the corresponding functional determinant. The appropriate basis is that of spinor spherical harmonics, and the eigenvalues¶ and degeneracies have been recently computed in connection with fermionic double-trace deformations [18]

$$\text{eigenvalues: } \pm \frac{\Gamma(l + n/2 + \nu + 1/2)}{\Gamma(l + n/2 - \nu + 1/2)} , \quad (15)$$

$$\text{degeneracies: } 2^{\lfloor \frac{n}{2} \rfloor} \frac{(l + n - 1)!}{l!(n - 1)!} . \quad (16)$$

|| Difference of the effective Lagrangians.

¶ The eigenvalues can be also read off from the scattering problem in H^{n+1} [32].

The formal trace is then assembled⁺ as follows:

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = 2^{1+\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\infty} \frac{(n)_l}{l!} \log \frac{\Gamma(l + \frac{n+1}{2} + \nu)}{\Gamma(l + \frac{n+1}{2} - \nu)}, \quad (17)$$

and it can be worked out within dimensional regularization as in [11, 21]

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = -2^{1+\lfloor \frac{n}{2} \rfloor} \Gamma(-n) \int_0^\nu d\mu \left\{ \frac{\Gamma(\frac{n+1}{2} + \mu)}{\Gamma(\frac{1-n}{2} + \mu)} + (\mu \rightarrow -\mu) \right\}. \quad (18)$$

This very same regularized answer we have already found in [16] (eqn.5) where Barnes's multiple gamma turned up. The renormalized value can be written, modulo polynomial and logarithmic terms (we refer again to [16] for further details and eqn.6), as the following quotient of Barnes' multiple gamma functions

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = 2^{1+\lfloor \frac{n}{2} \rfloor} \log \frac{\Gamma_{n+1}(\frac{n+1}{2} + \nu)}{\Gamma_{n+1}(\frac{n+1}{2} - \nu)}. \quad (19)$$

As explained in [16], the gamma factor in front of (18) is deceiving: only for n even there is a pole and from the residue one can read off the infinitesimal (integrated) conformal anomaly* under conformal rescaling of the metric

$$\frac{2^{2+\lfloor \frac{n}{2} \rfloor} (-1)^{n/2}}{n!} \int_0^\nu d\mu \left(\frac{1}{2} + \mu \right)^{\frac{n}{2}} \cdot \left(\frac{1}{2} - \mu \right)^{\frac{n}{2}}. \quad (20)$$

At this point we have a perfect match between bulk and boundary computations. To illustrate the usefulness of this result, besides being a explicit corroboration of the holographic formula, we study a particular value of the spinor mass that unveils the Dirac operator on the boundary and connects with a vast mathematical literature on determinants of differential operators on spheres (cf. [33]-[39]).

6. Holographic life of Dirac operator

There are two direct ways to identify the Dirac operator at the boundary.

- First, consider the action of the Dirac operator on the flat space two-point function (eqn. 3)

$$\nabla \langle \mathcal{O}_+(\vec{x}) \overline{\mathcal{O}}_+(\vec{0}) \rangle_{R^n} \sim \frac{\mathbb{I}}{|\vec{x}|^{n+1+m}}. \quad (21)$$

Here one can recognize the Laplacian ∇^2 in the limit $m \rightarrow \frac{1}{2}$, in a distributional sense.

- Second, by simple inspection of the eigenvalues of the kernel $\langle \mathcal{O}_+ \overline{\mathcal{O}}_+ \rangle_{S^n}$ on the n -sphere (eqn.15) in the same limit $m \rightarrow \frac{1}{2}$

$$\pm \left(\frac{n}{2} + l \right). \quad (22)$$

⁺ In dimensional regularization, the sum over degeneracies of a constant term vanishes, this is why we do not worry much about the fact that half of the eigenvalues are negative.

* The shorthand notation involves Pochhammer's symbol $(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}$.

6.1. Type A trace anomaly and Polyakov formulas

We are interested in the universal type A component of the trace anomaly. For conformally related metrics $\hat{g} = e^{2w}g$, in the conformal flat class, Branson conjectured the following Polyakov formula

$$-\log \frac{\det \hat{\mathcal{X}}^2}{\det \mathcal{X}^2} = c_{\mathcal{X}^2}^{(n)} \int_{\mathcal{M}} w \left(\hat{Q}_n dv_{\hat{g}} + Q_n dv_g \right) + \dots \quad (23)$$

This very same structure we read from the bulk computation, the Q-curvature terms come from the finite conformal variation of the renormalized volume [40, 41, 42] and the overall coefficient $c_{\mathcal{X}^2}^{(n)}$ is obtained from the effective Lagrangian at $m = \frac{1}{2}$

$$c_{\mathcal{X}^2}^{(n)} = 4 \frac{c_{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{\frac{1}{2}} d\nu \frac{\left(\frac{1}{2} + \nu\right)_{\frac{n}{2}} \cdot \left(\frac{1}{2} - \nu\right)_{\frac{n}{2}}}{\left(\frac{1}{2}\right)_{\frac{n}{2}}}, \quad (24)$$

where $c_k = \frac{(-1)^k}{2^{2k} k! (k-1)!}$. All values reported in [33] for $c_{\mathcal{X}^2}^{(n)}$ are correctly reproduced by this formula.

6.2. Determinant of iterated Dirac on spheres

The particular value of the determinant of the scattering operator at the mass value $\frac{1}{2}$ results in the following remarkable expression in terms of Barnes' multiple gamma function, after use of recurrence (A.3)

$$-\log \det \mathcal{X}^2 = 4 \cdot 2^{\lceil \frac{n}{2} \rceil} \cdot \log \Gamma_n \left(\frac{n}{2} \right). \quad (25)$$

This compact expression seems to correctly reproduce all zeta-regularized values reported in the literature (cf. [35, 39]).

A small digression on a conjecture put forward by Bär and Shopka [35]: they observed that the numerical values of these determinants tend to 1 as the dimension n grows; this was proved in [37] not only for the Dirac operator, but also for the Yamabe or conformal Laplacian on the n -sphere. Our result indicates that the above findings amount to establishing the limiting value of Barnes' gamma $\Gamma_n(n/2)$ as $n \rightarrow \infty$. Furthermore, we can interpret quite naturally this limiting value by looking at the bulk side of the holographic formula: the two boundary conditions coincide as n grows, or alternatively, $\lambda_{\pm} \rightarrow \frac{n}{2}$ so that both determinants in the quotient approach one another. This very same argument would predict the same limiting value for all other operators on the right side of similar holographic formulas; this is the case for the determinants of GJMS operators on round spheres [12]. In consequence, we predict the same limiting value of unity; a prediction that can be probed by the asymptotic analysis of the explicit results obtained in [38].

7. C-theorem proposals and entanglement entropy

Our computations contain and, at times, generalize several scattered results that had been independently derived, most of them in the pursuit of extensions of the C-theorem to higher dimensions and in connection with certain universal terms in entanglement entropy. We briefly list few of them.

Holographic C-charge at $\mathcal{O}(1)$:

- $n = \text{even}$, relative change at order $\mathcal{O}(1)$ in Cardy's central charge computed in [18]. $C_{UV} - C_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal *log-term* in entanglement entropy [20].

We find agreement between bulk and boundary outcomes - tables (1) and (2) in [18]- and our eqns. (14) and (20), respectively. Our calculation accounts for the overall coefficient as well.

- $n = \text{odd}$, relative change at order $\mathcal{O}(1)$ in one-loop effective action, candidate for central charge, computed in [18]. $C_{UV} - C_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal *constant term* in entanglement entropy [20].

We find agreement between bulk outcome -integral of eqn.(3.37) in [18]- and our eqn.(13). We are also able to fill in the 'hole' left in [18] by identifying this finite contribution on the CFT side, including the overall coefficient as well (eqn.19).

F-coefficient of odd-dimensional CFT's:

- F-coefficients for free massless Dirac fermions on odd-spheres, table (2) in [21].

This agrees with our result[‡] in eqn.(25) and, of course, with the values reported in [35, 37]. The decrease of the numerical values in the above table is precisely the hint for the conjecture by Bär and Schopka.

- Under RG flow triggered by a fermionic double-trace deformation, at leading large N , the change in free energy in three dimensions is given by eqn.(82),[21]. $F_{UV} - F_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$.

This coincides again with our bulk(eqn.13) and boundary(eqn.19) results.

Entanglement entropy:

- Universal terms in entanglement entropy [20]: logarithmic and constant in even and odd dimensions, respectively. In the case of a free boson, they coincide with the holographic anomaly and determinant of Yamabe or conformal Laplacian operator, for even [38] and odd [43] dimensions, respectively.

This matching holds as well for the entanglement entropy of a free Dirac spinor and the trace anomaly and determinant of the Dirac operator on n -spheres; now under the guise of holographic C-charge at $\mathcal{O}(1)$.

8. Conclusion

In all, we have worked out the spinor version of the holographic formula which connects functional determinants of bulk spinor with that of the two point function of the dual operator at the boundary. The case of pure AdS allows for explicit computations

[‡] The evaluation of the corresponding Barnes' gammas can be performed with the relations collected in Appendix A, and one can rewrite in terms of Riemann zeta by use of the relation $\zeta'(-2n) = \frac{(-1)^n (2n)!}{2^{1+2n} \pi^{2n}} \zeta(1+2n)$.

which, in turn, encompass several results independently derived in other contexts. In particular, contact is made in the case of the Dirac operator on the boundary. Here we have obtained a compact expression for the determinant on spheres in terms of Barnes' gamma function as well as the generic type-A trace anomaly in any even dimension. We have also gained insight into the conjecture by Bär and Schopka and unveiled its possible holographic roots.

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Appendix A. Barnes' multiple gamma: disambiguation

There are several choices for the normalization of Barnes' multiple gamma function $\Gamma_n(z)$. For definiteness, we stick to the convention in [39, 44, 45] and list few properties that are relevant to our calculations.

- Its logarithm can be written in terms of derivatives of Hurwitz zeta function

$$\log \Gamma_n(z) = \sum_{k=0}^{n-1} b_{n,k}(z) \cdot \zeta'(-k, z) , \quad (\text{A.1})$$

where $b_{n,k}(z)$ is a polynomial in z with Stirling numbers of the first kind $s(n, j)$ in its coefficients

$$b_{n,k}(z) = \frac{(-1)^{n-1-k}}{(n-1)!} \sum_{j=k}^{n-1} \binom{j}{k} \cdot s(n, j+1) \cdot z^{j-k} . \quad (\text{A.2})$$

- Recurrence or ladder relation:

$$\Gamma_{n+1}(1+z) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)} . \quad (\text{A.3})$$

- Pascal triangle by successive applications of the ladder relation:

$$\log \Gamma_n(m+z) = \sum_{l=0}^m (-1)^l \binom{m}{l} \cdot \log \Gamma_{n-l}(z) , \quad 0 \leq m \leq n-1 . \quad (\text{A.4})$$

- Particular values in terms of derivatives of Riemann zeta function^{††}:

$$\log \Gamma_n(1) = \sum_{k=0}^{n-1} b_{n,k}(1) \cdot \zeta'(-k) , \quad (\text{A.5})$$

$$\log \Gamma_n\left(\frac{1}{2}\right) = \sum_{k=0}^{n-1} b_{n,k}\left(\frac{1}{2}\right) \cdot (2^{-k} - 1) \cdot \zeta'(-k) - \log 2 \sum_{k=0}^{n-1} b_{n,k}\left(\frac{1}{2}\right) \frac{2^{-k} B_{k+1}}{k+1} , \quad (\text{A.6})$$

where B_n are Bernoulli numbers.

^{††}Relation A.5 can be further simplified to $\log \Gamma_n(1) = \frac{1}{(n-1)!} \sum_{j=1}^{n-1} |s(j, n-1)| \cdot \zeta'(-k)$, cf. [34] where their P_n is just the inverse of the gamma we are using.

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Determinant and Weyl anomaly of Dirac operator: a holographic derivation

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Abstract. We present a holographic formula relating functional determinants: the fermion determinant in the one-loop effective action of bulk spinors in an asymptotically locally AdS background, and the determinant of the two-point function of the dual operator at the conformal boundary. The formula originates from AdS/CFT heuristics that map a quantum contribution in the bulk partition function to a subleading large- N contribution in the boundary partition function. We use this holographic picture to address questions in spectral theory and conformal geometry. As an instance, we compute the type-A Weyl anomaly and the determinant of the iterated Dirac operator on round spheres, express the latter in terms of Barnes' multiple gamma function and gain insight into a conjecture by Bär and Schopka.

1. Introduction

Ever since its appearance almost fifteen years ago in the form of Maldacena's conjecture, the AdS/CFT correspondence [1, 2, 3] has been a successful tool to address questions concerning strongly-coupled systems. Many developments depart from the original canonical formulation in pure anti-de Sitter (AdS) spacetime (mostly still restricted to classical (super)gravity in the bulk); for instance, finite-temperature effects on the boundary theory led to consideration of AdS black holes as bulk background geometries. This novel holographic approach, phenomenological in nature, covers an increasing amount of physical situations ranging from strongly coupled quark-gluon plasma and condensed matter systems to cosmological singularities and black hole physics. At present, this ambitious but conjectural program seems to succeed at a qualitative level (cf. [4]).

In contrast, quantitative and exact results in AdS/CFT correspondence are to be found in the interplay with mathematics; in for instance conformal geometry and spectral theory. Geometric roots of AdS/CFT date back to the seminal work of Fefferman and Graham [5, 6] that addresses conformal geometry on a compact manifold as geometry at the conformal infinity of space-filling Poincare metrics. Conversely, AdS/CFT revealed interesting conformal invariants, e.g. Q -curvature, that arise in the volume renormalization of these Poincare metrics and triggered new developments in conformal geometry (cf. [7, 8]).

The present contribution will focus precisely on these latter aspects of the duality, where the foreseeable progress seems modest but solid. We deal with 'holographic formulas' as special entries in the AdS/CFT dictionary, relating one-loop

determinants for bulk fields in asymptotically AdS backgrounds and determinants of correlation functions of the dual operators at the boundary. They originate in quantum refinements of the duality where one-loop corrections in the gravity side are mapped to sub-leading terms in the large-N expansion of the boundary theory. Interesting effects in for instance thermodynamics and transport phenomena on the boundary are captured by the holographic correspondence only after inclusion of quantum one-loop effects in the bulk (cf. [9, 10]). A systematic study of bulk *scalars* has led to a holographic formula which has been verified in certain cases amenable to analytic evaluation; these bulk geometries include pure and thermal AdS, the BTZ black hole, and other quotients or orbifolds of AdS [11, 12, 13, 14].

Our aim now is to show that also for bulk *spinors* an analogous holographic formula can be established. Explicit computations are performed for the ball model of hyperbolic space which embrace several scattered results in the literature. In this case the bulk side and the role of Barnes' gamma function, already explored in [15, 16], can be further exploited to get a closed formula for the determinant of the Dirac operator on round spheres, an interesting result that seems to have escaped notice. A universal formula for the type-A trace anomaly [17] of Dirac operator is as well obtained in this holographic way. These explicit results contain previous ones found in relation with proposals for a c-theorem in dimensions other than two; they include Cardy's a-theorem [18, 19], universal terms in entanglement entropy [20, 21] and F-theorem [22].

We start in section 2 with a review of bulk spinors in AdS/CFT and its double quantization to predict an $O(1)$ quantum contribution to the partition functions. Next we analyze the dual picture at the boundary in section 3 in order to detect this $O(1)$ contribution to the partition function on the CFT side. In section 4 we write down the spinor holographic formula. In section 5, the pure AdS bulk geometry is considered as an instance where both sides of the formula can be worked out in detail. Section 6 is concerned with the Dirac operator at the boundary and the application of the holographic formula to read off the universal part in the associated Polyakov formulas, the type-A trace anomaly as well as the functional determinant on round spheres. In section 7 we examine several scattered results closely related to our calculations. Concluding remarks are given in section 8, and conventions and useful identities for Barnes' gammas are collected in an appendix.

2. Bulk spinors and double quantization

The role of bulk spinors in AdS/CFT has, of course, been extensively studied since the early days of the correspondence; a non-exhaustive list includes [23, 24, 25, 26]. To begin with, we choose the bulk side and review the features relevant to our present concern, namely, the spinor version of the holographic formula.

Consider a bulk metric that approaches asymptotically the Poincaré half-space model for the Euclidean section of AdS_{n+1} , that is, hyperbolic space H^{n+1}

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} . \quad (1)$$

The solutions of Dirac equation $(\not{X} + m)\psi = 0$, with positive mass $m > 0$ for definiteness, behave near the conformal boundary $z = 0$ as[‡]

$$\psi \sim z^{\lambda_-} \psi_o(\vec{x}) + z^{\lambda_+} \chi_o(\vec{x}) , \quad (2)$$

[‡] Strictly speaking, valid for $0 < m < \frac{1}{2}$.

with $\lambda_{\pm} = \frac{n}{2} \pm m$, and the boundary data ψ_o and χ_o belong to the eigenspace of the flat Dirac gamma associated to the z -direction, Γ^o , with eigenvalues -1 and $+1$, respectively.

The requirement of regularity at the deep interior, $z \rightarrow \infty$, imposes a linear relation between ψ_o and χ_o given by convolution with the scattering operator $\chi_o = S(\lambda) * \psi_o$, or equivalently, with the kernel associated to the two-point function of the dual operator at the boundary. It is this on-shell relation which ultimately leads, upon functional differentiation of the action with respect to the boundary source, to the corresponding two-point correlator

$$\langle \mathcal{O}_+ \bar{\mathcal{O}}_+ \rangle \sim \frac{\vec{\Gamma} \cdot (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{1+n+2m}} . \quad (3)$$

The standard AdS/CFT recipe contemplates ψ_o as the source and χ_o as expectation value of the dual primary operator with conformal dimension λ_+ at the boundary. In the variational approach to the path integral, the sum over histories is preformed with ψ_o and $\bar{\psi}_o$ prescribed, whereas χ_o and $\bar{\chi}_o$ are free to vary. Nonetheless, a crucial observation (cf. [19, 22, 27]) is that whenever $0 < m < 1/2$ the dimension λ_- is above the unitarity bound, in this case one is free to interchange the roles of ψ_o and χ_o to obtain the two-point function of a dual operator of dimension λ_-

$$\langle \mathcal{O}_- \bar{\mathcal{O}}_- \rangle \sim \frac{\vec{\Gamma} \cdot (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{1+n-2m}} . \quad (4)$$

2.1. $O(1)$ contribution to the partition function

That is very much the picture at the classical level in the bulk. Considering now quantum fluctuations, we go off-shell and there are two possible AdS-invariant quantizations of the bulk spinor: the conventional one with ψ_o set to zero and another, ‘alternate’ one, with χ_o set to zero. Whenever the mass of the bulk spinor lies in the window $0 < m < \frac{1}{2}$, both kinds of modes are normalizable (finite energy configurations [27]). Their contribution to the partition function can be computed in the standard way via the Green function for the conventional modes (λ_+), and analytic continuation to account for the alternate modes (λ_-). With this choice at hand, we can emulate the scalar case [28] since now double quantization for bulk spinors is established. The relative change in the partition function, upon functional integration of the quantum fluctuations at quadratic order, is then given by the ratio of the associated functional determinants:

$$\frac{Z_{grav}^+}{Z_{grav}^-} = \frac{\det_+ \{ \not{\nabla}_x + m \}}{\det_- \{ \not{\nabla}_x + m \}} . \quad (5)$$

3. Boundary double-trace deformation

These two choices of asymptotic behavior correspond in AdS/CFT to two CFT’s [27, 29] that share the same field content but differ in the dimension of the fermionic operator \mathcal{O} , dual to the bulk spinor. Despite its appearance, the situation is by no means symmetric; the UV CFT with \mathcal{O}_- , perturbed by the *relevant* double-trace deformation \mathcal{O}_-^2 , flows into the IR CFT with \mathcal{O}_+ (cf. [19, 22, 27]).

3.1. $O(1)$ contribution to the partition function: a shortcut

To get a handle on the relative change in the CFT partition functions at the end points of the RG flow, instead of considering the auxiliary field trick as in [19, 22], we simply adapt the heuristic argument given in [30] to relate Z_{UV} to Z_{IR} . Namely, in the path integral of the UV CFT we promote the sources η and $\bar{\eta}$ to dynamical fields and integrate over them

$$\frac{Z_{IR}}{Z_{UV}} = \int \mathcal{D}\bar{\eta} \mathcal{D}\eta \langle \exp \int (\bar{\eta} \mathcal{O}_- + \bar{\mathcal{O}}_- \eta) \rangle . \quad (6)$$

The expectation value can be approximated, at leading large N due to the factorization of the correlation functions, by

$$\exp \int \bar{\eta} \langle \mathcal{O}_- \bar{\mathcal{O}}_- \rangle \eta , \quad (7)$$

and the Gaussian integral results in the functional determinant of the two-point function

$$\frac{Z_{IR}}{Z_{UV}} = \det \langle \mathcal{O}_- \bar{\mathcal{O}}_- \rangle , \quad (8)$$

or, alternatively,

$$\frac{Z_{UV}}{Z_{IR}} = \det \langle \mathcal{O}_+ \bar{\mathcal{O}}_+ \rangle . \quad (9)$$

4. The holographic formula

We have now all necessary ingredients to write down the ‘spinor holographic formula’ that stems from the postulated equality of the partition functions in AdS/CFT correspondence, at subleading order $O(1)$:

$$\frac{\det_- \{\nabla_X + m\}}{\det_+ \{\nabla_X + m\}} = \det \langle \mathcal{O}_\lambda \bar{\mathcal{O}}_\lambda \rangle_{\mathcal{M}} \quad (10)$$

The ‘+’ means that we compute with the standard λ in the asymptotic behavior of the bulk spinor, whereas ‘−’ means the analytic continuation $\lambda \rightarrow n - \lambda$. As usual, to make sense out of this formula one needs to tame the divergencies that arise due to the IR divergent volume of AdS and the UV divergent short distance singularities. Both divergencies turn out to be tied by the IR/UV connection in AdS/CFT correspondence [31].

This formula conjecturally applies to bulk geometries X which are Euclidean sections of asymptotically locally AdS (ALAdS). In particular, when the conformal infinity \mathcal{M} belongs to the conformal class of the standard round spheres, and therefore conformally flat, the bulk is locally AdS and the IR-divergent volume of AdS factorizes. One can then read off an $O(1)$ contribution to the holographic trace anomaly in even n , just as in the case of a bulk scalar [3, 7, 32]. Alternatively, from the difference of one-loop effective actions one can compute the holographic type-A trace anomaly coefficient a following the general recipe of [33]. The behavior of this coefficient for even n and of a related quantity for odd n , in the cases we will explore, gives support to a conjectured c-theorem valid in all dimensions [21] § and to an F-theorem proposal [22] as well.

§ This promises to settle the disparity pointed out in [34] that, although the computation on the AdS side contemplates even and odd dimensions on equal footing, it is not clear how to translate the “holographic” central charge into field theory language in the case of odd-dimensional CFT’s.

5. The canonical case

As might be expected, the ball model for hyperbolic space (Euclidean AdS) turns out to be the simplest bulk background where calculations can be spelled out in detail and related to the CFT on the conformal boundary (the conformally flat class of the standard round sphere).

5.1. Bulk

The effective action for a Dirac spinor in hyperbolic space has been recently revisited [15, 16] in connection with a curious gauge-gravity duality where Barnes' multiple gamma function plays a central role. We briefly survey the relevant steps in the computation of the effective action

$$S_{grav}^+ = -\log \det\{\not{N} + m\} . \quad (11)$$

In terms of the Green's function, one has $(\not{N} + m)\mathcal{D} = -\mathbb{I}$,

$$S_{grav}^+ = \int^m \text{tr } \mathcal{D}^{(n+1)} , \quad (12)$$

where also the spinor indices are traced out. There is a subtlety regarding the dimensionality of the representations of the gamma matrices: for n odd, bulk and boundary representations share the same dimensionality; whereas for n even, in order to have a Dirac fermion on the boundary, the dimensionality of the bulk representation must be doubled [27].

We refer to [15, 16] for details of the implementation of dimensional regularization and the nontrivial role of the bulk volume. In all, one gets a remarkable result, valid for both even and odd dimensions, in terms of Barnes' multiple gamma

$$\log \frac{\det_+ \{\not{N} + m\}}{\det_- \{\not{N} + m\}} = -2^{1+\lfloor \frac{n}{2} \rfloor} \cdot \log \frac{\Gamma_{n+1}(\frac{n+1}{2} + m)}{\Gamma_{n+1}(\frac{n+1}{2} - m)} . \quad (13)$$

In addition, whenever n is even one can read off the trace anomaly as in the scalar case [11, 12]. In the present case one essentially gets the integral of the spinor Plancherel|| measure(cf. [35]) times the volume anomaly $\mathcal{L}_{n+1} = 2(-\pi)^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$

$$\left[\frac{2}{(2\pi)^{\frac{n}{2}}} \int_0^m d\mu \frac{(\frac{1}{2} + \mu)^{\frac{n}{2}} \cdot (\frac{1}{2} - \mu)^{\frac{n}{2}}}{(\frac{1}{2})^{\frac{n}{2}}} \right] \cdot \mathcal{L}_{n+1} . \quad (14)$$

5.2. Boundary

For the round n -sphere as conformal boundary, the knowledge of the eigenvalues of the two-point function $\langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n}$ and their degeneracies allows a brute-force computation of the corresponding functional determinant. The appropriate basis is that of spinor spherical harmonics, and the eigenvalues¶ and degeneracies have been recently computed in connection with fermionic double-trace deformations [19]

$$\text{eigenvalues: } \pm \frac{\Gamma(l + n/2 + \nu + 1/2)}{\Gamma(l + n/2 - \nu + 1/2)} , \quad (15)$$

|| Difference of the effective Lagrangians. The shorthand notation involves Pochhammer's symbol $(x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}$.

¶ The eigenvalues can be also read off from the scattering problem in H^{n+1} [36].

$$\text{degeneracies: } 2^{\lfloor \frac{n}{2} \rfloor} \frac{(l+n-1)!}{l!(n-1)!} . \quad (16)$$

The formal trace is then assembled⁺ as follows:

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = 2^{1+\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\infty} \frac{(n)_l}{l!} \log \frac{\Gamma(l + \frac{n+1}{2} + \nu)}{\Gamma(l + \frac{n+1}{2} - \nu)} , \quad (17)$$

and it can be worked out within dimensional regularization as in [11, 22]

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = -2^{1+\lfloor \frac{n}{2} \rfloor} \Gamma(-n) \int_0^\nu d\mu \left\{ \frac{\Gamma(\frac{n+1}{2} + \mu)}{\Gamma(\frac{1-n}{2} + \mu)} + (\mu \rightarrow -\mu) \right\} . \quad (18)$$

This very same regularized answer we have already found in [16], (eqn.5), where Barnes's multiple gamma turned up. The renormalized value can be written, modulo polynomial and logarithmic terms (we refer again to [16] for further details and eqn.6), as the following quotient of Barnes' multiple gamma functions

$$\log \det \langle \mathcal{O}_\lambda \overline{\mathcal{O}}_\lambda \rangle_{S^n} = 2^{1+\lfloor \frac{n}{2} \rfloor} \log \frac{\Gamma_{n+1}(\frac{n+1}{2} + \nu)}{\Gamma_{n+1}(\frac{n+1}{2} - \nu)} . \quad (19)$$

As explained in [16], the gamma factor in front of (18) is deceiving: only for n even there is a pole and from the residue one can read off the (integrated) conformal anomaly under conformal rescaling of the metric

$$\frac{2^{2+\frac{n}{2}} (-1)^{n/2}}{n!} \int_0^\nu d\mu \left(\frac{1}{2} + \mu \right)^{\frac{n}{2}} \cdot \left(\frac{1}{2} - \mu \right)^{\frac{n}{2}} . \quad (20)$$

At this point we have a perfect match between bulk and boundary computations. To illustrate the usefulness of this result, besides being an explicit corroboration of the holographic formula, we study a particular value of the spinor mass that unveils the Dirac operator on the boundary and connects with a vast mathematical literature on determinants of differential operators on spheres (cf. [37]-[43]).

6. Holographic life of Dirac operator

There are two direct ways to identify the Dirac operator at the boundary.

- First, consider the action of the Dirac operator on the flat space two-point function (eqn. 3)

$$\nabla \langle \mathcal{O}_+(\vec{x}) \overline{\mathcal{O}}_+(\vec{0}) \rangle_{R^n} \sim \frac{\mathbb{I}}{|\vec{x}|^{n+1+2m}} . \quad (21)$$

Here one can recognize the Laplacian ∇^2 in the limit $m \rightarrow \frac{1}{2}$, in a distributional sense.

- Second, by simple inspection of the eigenvalues of the kernel $\langle \mathcal{O}_+ \overline{\mathcal{O}}_+ \rangle_{S^n}$ on the n -sphere (eqn.15) in the same limit $m \rightarrow \frac{1}{2}$

$$\pm \left(\frac{n}{2} + l \right) . \quad (22)$$

⁺ In dimensional regularization, the sum over degeneracies of a constant term vanishes, this is why we do not worry much about the fact that half of the eigenvalues are negative.

6.1. Type-A trace anomaly and Polyakov formulas

We are interested in the universal type-A component of the trace anomaly, the coefficient of the Euler density or Pfaffian according to the classification in [17]. For conformally related metrics $\hat{g} = e^{2w}g$, in the conformally flat class, Branson (see, e.g., [37]) conjectured the following Polyakov-like formula

$$-\log \frac{\det \hat{\mathbb{X}}^2}{\det \mathbb{X}^2} = c_{\mathbb{X}^2}^{(n)} \int_{\mathcal{M}} w \left(\hat{Q}_n dv_{\hat{g}} + Q_n dv_g \right) + \dots, \quad (23)$$

where the Pfaffian is traded by Branson's Q-curvature Q_n , a central object in conformal geometry with a much simpler (in fact, linear in w) transformation rule under conformal rescalings.

This very same structure we read from the bulk computation, the Q-curvature terms come from the finite conformal variation of the renormalized volume [44, 45, 46] and the overall coefficient $c_{\mathbb{X}^2}^{(n)}$ is obtained from the effective Lagrangian at $m = \frac{1}{2}$

$$c_{\mathbb{X}^2}^{(n)} = 4 \frac{c_{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{\frac{1}{2}} d\nu \frac{\left(\frac{1}{2} + \nu\right)^{\frac{n}{2}} \cdot \left(\frac{1}{2} - \nu\right)^{\frac{n}{2}}}{\left(\frac{1}{2}\right)^{\frac{n}{2}}}, \quad (24)$$

where $c_k = \frac{(-1)^k}{2^{2k} k! (k-1)!}$. All values reported in [37] for $c_{\mathbb{X}^2}^{(n)}$ are correctly reproduced by this formula. Interesting spectral invariants, as zeta function at zero argument and conformal index (cf. [37, 12]), are easily obtained from this coefficient.

6.2. Determinant of iterated Dirac on spheres

The particular value of the determinant of the scattering operator at the mass value $\frac{1}{2}$ results in the following remarkable expression in terms of Barnes' multiple gamma function, after use of recurrence (A.3),

$$-\log \det \mathbb{X}^2 = 4 \cdot 2^{\lfloor \frac{n}{2} \rfloor} \cdot \log \Gamma_n \left(\frac{n}{2} \right). \quad (25)$$

Barnes' multiple gamma function is known to occur in functional determinants of Laplacians on spheres (see, e.g., references in [40]). And yet this compact expression, which correctly reproduces all zeta-regularized values reported in the literature (cf. [39, 43]), does not seem to have been noticed until now.

A small digression on a conjecture put forward by Bär and Shopka [39]: they observed that the numerical values of these determinants tend to 1 as the dimension n grows; this was proved in [41] not only for the Dirac operator, but also for the Yamabe or conformal Laplacian on the n -sphere. Our result indicates that the above findings amount to establishing the limiting value of Barnes' gamma $\Gamma_n(n/2)$ as $n \rightarrow \infty$. Furthermore, we can interpret quite naturally this limiting value by looking at the bulk side of the holographic formula: the two boundary conditions coincide as n grows, or alternatively, $\lambda_{\pm} \rightarrow \frac{n}{2}$ so that both determinants in the quotient approach one another. This very same argument would predict the same limiting value for all other operators on the right side of similar holographic formulas; this is the case for the determinants of GJMS operators on round spheres [12]. In consequence, we predict the same limiting value of unity; a prediction that can be probed by the asymptotic analysis of the explicit results obtained in [42] (see eqn.19 therein). In fact, the quotient of Barnes's gammas in the limit of large dimension d and fixed order k tends to 1, as was the case for the Dirac operator; at the same time, the reported numerical

values for the multiplicative anomaly $M(d, k)$ hint at a vanishing limit value, so that the exponential of both contributions in eqn.(19) of [42] should again render 1 in the limit.

7. C-theorem proposals and entanglement entropy

Our computations contain and, at times, generalize several scattered results that had been independently derived, most of them in the pursuit of extensions of the C-theorem to higher dimensions and in connection with certain universal terms in entanglement entropy. We briefly list few of them.

Holographic C-charge at $\mathcal{O}(1)$:

- $n = \text{even}$, relative change at order $\mathcal{O}(1)$ in Cardy's central charge computed in [19]. $C_{UV} - C_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal *log-term* in entanglement entropy [21].

We find agreement between bulk and boundary outcomes - tables (1) and (2) in [19]- and our eqns. (14) and (20), respectively. Our calculation accounts for the overall coefficient as well.

- $n = \text{odd}$, relative change at order $\mathcal{O}(1)$ in one-loop effective action, candidate for central charge, computed in [19]. $C_{UV} - C_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal *constant term* in entanglement entropy [21].

We find agreement between bulk outcome -integral of eqn.(3.37) in [19]- and our eqn.(13). We are also able to fill the 'hole' left in [19] by identifying this finite contribution on the CFT side, including the overall coefficient as well (eqn.19).

F-coefficient of odd-dimensional CFT's:

- F-coefficients for free massless Dirac fermions on odd-spheres, table (2) in [22].

This agrees with our result* in eqn.(25) and, of course, with the values reported in [39, 41]. The decrease of the numerical values in the above table is precisely the hint for the conjecture by Bär and Schopka.

- Under RG flow triggered by a fermionic double-trace deformation, at leading large N, the change in free energy in three dimensions is given by eqn.(82),[22]. $F_{UV} - F_{IR} > 0$ in the mass window $0 < m \leq \frac{1}{2}$.

This coincides again with our bulk (eqn.13) and boundary (eqn.19) results.

Entanglement entropy:

- Universal terms in entanglement entropy [21]: logarithmic and constant in even and odd dimensions, respectively. In the case of a free boson, they coincide with the holographic anomaly and determinant of Yamabe operator or conformal

* The evaluation of the corresponding Barnes' gammas can be performed with the relations collected in Appendix A, and one can rewrite in terms of Riemann zeta by use of the relation $\zeta'(-2n) = \frac{(-1)^n (2n)!}{2^{1+2n} \pi^{2n}} \zeta(1+2n)$.

Laplacian, for even [42] and odd [47] dimensions, respectively.

This matching should hold as well for the entanglement entropy of a free Dirac spinor and the trace anomaly and determinant of the Dirac operator on n -spheres. Now under the guise of holographic C-charge at $\mathcal{O}(1)$, in the notation of [21], they verify $(a_d^*)_{UV} > (a_d^*)_{IR}$.

8. Conclusion

Our main contribution has been the spinor version of the holographic formula that connects the functional determinant of a bulk spinor with that of the two point function of the dual operator at the boundary. The case of pure AdS allows for explicit computations which, in turn, encompass several results independently derived in other contexts. In particular, contact is made in the case of the Dirac operator on the boundary; here we have obtained a compact expression for the determinant on spheres in terms of Barnes' gamma function as well as the generic type-A trace anomaly in any even dimension.

We have also gained insight into the conjecture by Bär and Schopka and unveiled its possible holographic roots. It seems plausible that the conjecture should also apply to the functional determinant of *any* conformally covariant differential operator, provided a corresponding holographic formula exists. Natural candidates for the bulk side of such holographic formula are the functional determinants in the one-loop effective action of higher-spin fields (see, e.g., recent progress in [48]).

Further study of the holographic formula, for instance in black-hole backgrounds, remains a challenge. It seems worth to explore the connection of the functional determinants involved in the formula with regularized products of quasinormal frequencies [9] or scattering resonances. Another possible avenue concerns an intriguing feature of Barnes' double gamma at integer values, it equals the result of ordinary determinants of Hankel matrices; this might well be a sign for *localization* of the functional integrals that we have been computing in AdS (see, in this respect, [49]).

We close by noticing that the explicit expressions, as obtained via the holographic approach, come out in a gracious, simple form. This was the case for the trace anomaly of GJMS operators [12] and now for the determinant and trace anomaly of Dirac operator.

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Appendix A. Barnes' multiple gamma: disambiguation

There are several choices for the normalization of Barnes' multiple gamma function $\Gamma_n(z)$. For definiteness, we stick to the convention in [43, 50, 51] and list few properties that are relevant to our calculations.

- Its logarithm can be written in terms of derivatives of Hurwitz zeta function

$$\log \Gamma_n(z) = \sum_{k=0}^{n-1} b_{n,k}(z) \cdot \zeta'(-k, z) , \quad (\text{A.1})$$

where $b_{n,k}(z)$ is a polynomial in z with Stirling numbers of the first kind $s(n, j)$ in its coefficients

$$b_{n,k}(z) = \frac{(-1)^{n-1-k}}{(n-1)!} \sum_{j=k}^{n-1} \binom{j}{k} \cdot s(n, j+1) \cdot z^{j-k} . \quad (\text{A.2})$$

- Recurrence or ladder relation:

$$\Gamma_{n+1}(1+z) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)} . \quad (\text{A.3})$$

- Pascal triangle by successive applications of the ladder relation:

$$\log \Gamma_n(m+z) = \sum_{l=0}^m (-1)^l \binom{m}{l} \cdot \log \Gamma_{n-l}(z) , \quad 0 \leq m \leq n-1 . \quad (\text{A.4})$$

- Particular values in terms of derivatives of Riemann zeta function[‡]:

$$\log \Gamma_n(1) = \sum_{k=0}^{n-1} b_{n,k}(1) \cdot \zeta'(-k) , \quad (\text{A.5})$$

$$\log \Gamma_n\left(\frac{1}{2}\right) = \sum_{k=0}^{n-1} b_{n,k}\left(\frac{1}{2}\right) \cdot (2^{-k} - 1) \cdot \zeta'(-k) - \log 2 \sum_{k=0}^{n-1} b_{n,k}\left(\frac{1}{2}\right) \frac{2^{-k} B_{k+1}}{k+1} , \quad (\text{A.6})$$

where B_n are Bernoulli numbers.

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[‡] Relation A.5 can be further simplified to $\log \Gamma_n(1) = \frac{1}{(n-1)!} \sum_{j=1}^{n-1} |s(j, n-1)| \cdot \zeta'(-k)$, cf. [38] where their P_n is just the inverse of the gamma we are using.

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